

Universality of the category of schemes

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Abstract

In this paper, we generalize the construction method of schemes to other algebraic categories, and show that the category of coherent schemes can be characterized by a universal property, if we fix the class of Grothendieck topology. Also, we introduce the notion of \mathcal{C} -schemes, which is a further generalization of coherent schemes and still shares common properties with ordinary schemes.

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1 Introduction

In this paper, we generalize the construction of schemes to other algebraic categories.

There are already many attempts of extending the category of schemes, for various reasons. One of it is from the geometry over \mathbb{F}_1 : the rough idea is to construct a scheme-like topological object from monoids. There are numerous attempts towards this goal, and we cannot list them all. Here, we just mention that there is a survey on this topic [PL]. In this article, we focus on the relation between the construction of ours and those of Toën-Vaquié [TV], and Deitmar [D1]. (the latter two constructions are actually equivalent, according to [PL].)

However, not many have mentioned its universal property of the proposed new objects and categories. To claim that the new objects are “good”, we must make it explicit how it is similar to schemes. In other words, we should ask to ourselves,

“What is a scheme, anyway?”

We know how schemes are useful, but we don’t know why. Despite its heavy and complicated machinery, the category of schemes behaves surprisingly good. This implies that the construction of schemes, or almost equivalently, the spectrum functor, should be realized as an adjoint of a simpler functor.

In this paper, we go back to this fundamental question: we give the universal characterization of the category of coherent schemes.

The universality cannot be obtained, only by specifying the algebraic object: even if we restrict ourselves to rings, there are several ways of defining a scheme-like objects. Therefore, we must designate its Grothendieck topology.

Also, we must be aware that the category of schemes is not complete, although the category of rings is co-complete. Therefore, in order to give a universal property, we should be able to apply the spectrum functor to schemes: if we call the scheme-like V -valued spaces as V -schemes, then we must be able to define $(V\text{-}\mathbf{Scheme})^{\text{op}}$ -schemes.

This implies that the usual definition of schemes is not appropriate: it uses the notion of local rings and local homomorphisms where infinite operations occur, which is not available for schemes in general. Therefore, we give another way of describing this local property, only using finite operations on the value category. This idea is already mentioned in the previous preprint

of the author [T1]. The advantage of this definition is that we can define the spectrum functor as an adjoint of the global section functor, which is by nature, much simpler. Another advantage is that this gives a way to define a larger category containing all coherent schemes, which is complete: see [T2].

Here, we will list up the main results. Let \mathcal{C} be a category with pull backs and finite coproducts, equipped with a Grothendieck topology $(\mathcal{C}, \mathcal{O})$ (the precise definition is given in §3). We call the triple $(\mathcal{C}, \mathcal{E}, \mathcal{O})$ a coherent topological category. This is *schematic*, if it admits finite open patchings (again, the definition is given in §3).

Theorem 1.1. Let **(CT-Cat)**, **(CT-Sch)** be the (2-)categories of coherent topological categories, schematic coherent topological categories, respectively.

- (1) The underlying functor $U : (\mathbf{CT-Sch}) \rightarrow (\mathbf{CT-Cat})$ admits a left adjoint $\langle \underline{\text{Sch}} \rangle$. The unit is the spectrum functor $\text{Spec} : \mathcal{C} \rightarrow (\mathcal{C}\text{-Sch}) = \langle \underline{\text{Sch}} \rangle(\mathcal{C})$. (Theorem 3.6)
- (2) When \mathcal{C} is the opposite category of commutative rings, with the topology induced from ideals, then \mathcal{C} -schemes are coherent schemes. (Theorem 4.14)

The universal property of the ideal topology is summarized as follows:

Theorem 1.2 (Theorem 5.6). Let V be a self enhancing algebraic type with a constant operator, and W be the type of commutative monoid objects in the category of V -algebras. If the morphism class \mathcal{E} consists of localizations of finite type, then the ideal topology is the coarsest topology satisfying the following condition:

For any W -algebra R and R -module M , the induced $\mathcal{O}_{\text{Spec } R}$ -module \tilde{M} is zero if and only if $M = 0$.

Note that this paper is only a stepping stone to various generalizations. However, we decided to rewrite it from the first step: since although the scheme theory is already a classical topic, its universal property is scarcely discussed, and we need it to be clarified to apply its machinery to other workfields.

This paper is organized as follows: in section §2, we give a brief summary of algebraic types and lattice theories. The reader may skip this section, if he or she is familiar with the terminology.

In §3, we discuss the universal property of \mathcal{C} -schemes: namely it is given by the universal property of schematic topological category under \mathcal{C} . In the process, we introduce the notion of weak \mathcal{C} -schemes, which is the generalization of \mathcal{A} -schemes. Using this, we obtain the spectrum functor as an adjoint of the global section functor.

In §4, we compare our notion of \mathcal{C} -schemes with the conventional coherent schemes. This is somewhat time-consuming, since the definitions of the two are very different. Here, we also show the correspondence between the ideals and the Grothendieck topology.

Finally in §5, we discuss the universal property of ideal topology. Note that the ground of choosing the ideal topology is only given by looking at modules. When we talk of algebraic objects other than rings, we know that congruences take place of ideals. However, the result obtained here shows that this is a groundless fear, since the ideal topology detects modules similarly as in the case of rings.

Notation and Conventions: In the sequel, we assume the reader to be familiar with category-theoretic languages; the textbook [CWM] will be sufficient for this purpose. We fix a universe, and do not mention it when unnecessary.

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2 Preliminaries

2.1 Algebras

Definition 2.1. (1) Let (\mathbf{Cat}) be the category of small categories. We have a natural functor $\text{obj} : (\mathbf{Cat}) \rightarrow (\mathbf{Set})$, sending a small category \mathcal{C} to its object $\text{obj}(\mathcal{C})$.

(2) Let (\mathbf{Cat}^\times) be the category of small categories with finite products; in particular, any small category $\mathcal{C} \in (\mathbf{Cat}^\times)$ has a terminal object 0 . We have a left adjoint $\langle \underline{\text{free}} \rangle$ of the underlying functor $\text{obj} : (\mathbf{Cat}^\times) \rightarrow (\mathbf{Set})$, and we set $\mathcal{X} = \langle \underline{\text{free}} \rangle(*)$. We denote by $[n]$ the n -product of $*$ in \mathcal{X} .

(3) An (*finitary*) *algebraic type* is a pointed category $V \in (\mathbf{Cat}^\times)$ with finite products such that

- (a) The induced map $\text{obj}(\mathcal{X}) \rightarrow \text{obj}(V)$ is bijective, and
- (b) the morphisms of V is generated by the morphisms of \mathcal{X} and a set Ω of morphisms of type $[n] \rightarrow [1]$, called the *n-ary operators*, divided by the equivalence relation $E \subset \text{Mor}(V)^2$, called the *equational class*. A *constant equation* is a relation (f, g) in E such that either f or g factors through $[0]$.
 V is denoted by $\langle \Omega, E \rangle$, and the elements of $V([n], [1])$ is called *n-ary derived operators*.

(4) A morphism $F : V \rightarrow W$ of algebraic types is a finite continuous functor of pointed categories.

(5) For any algebraic type V and an augmented symmetric monoidal category (\mathcal{C}, \otimes) , a *V-algebra* in \mathcal{C} is a morphism $A : V \rightarrow \mathcal{C}$ of monoidal categories. When the generating set of the equational class E does not contain any constant equation, then we can omit the assumption that \mathcal{C} is augmented. We often just indicate $A([1]) \in \mathcal{C}$ for A , if there seems to be no ambiguity.

A *morphism* $f : A \rightarrow B$ of *V-algebras* in \mathcal{C} is a \otimes -preserving natural transformation.

(6) We denote by $(V\text{-}\mathbf{alg}/\mathcal{C})$ the category of *V-algebras* in \mathcal{C} . When \mathcal{C} is the category (\mathbf{Set}) of small sets, regarded as a cartesian closed category, then we omit the indication of \mathcal{C} .

Definition 2.2. (1) Let $F : \mathcal{X} \rightleftarrows \mathcal{A} : U$ be an adjunction between two categories, and set $T = UF : \mathcal{X} \rightarrow \mathcal{X}$. Let \mathcal{X}^T be the category of *T-algebras*, namely objects X in \mathcal{X} with a *T-action* $TX \rightarrow X$ and *T-equivariant* morphisms. We have a natural functor $K : \mathcal{A} \rightarrow \mathcal{X}^T$, called the *comparison functor*. The above adjoint is *monadic*, if K is an equivalence of categories.

(2) A *finitary algebraic category* is a category \mathcal{A} equipped with a adjunction $(\mathbf{Set}) \rightleftarrows \mathcal{A}$ which is monadic.

Let $f : V \rightarrow W$ be a morphism of algebraic types. Then, f induces an underlying functor $U : (W\text{-}\mathbf{alg}) \rightarrow (V\text{-}\mathbf{alg})$ which has a left adjoint $\langle \underline{\text{free}} \rangle$. In particular, we have an adjoint $\langle \underline{\text{free}} \rangle : (\mathbf{Set}) \rightleftarrows (V\text{-}\mathbf{alg})$ for any algebraic type V , and this is monadic. Consequently, $(V\text{-}\mathbf{alg})$ is small complete and small co-complete. We refer to the functor $\langle \underline{\text{free}} \rangle$ as the *free generator*.

Remark 2.3. The definition of algebras can also be described using the notion of operads [Mm], but we will not discuss it here.

2.2 Lattice theories

In this section, we give a brief summary of lattice theories. This also emphasizes the fact that restricting to coherent schemes is somewhat better than considering all schemes.

Definition 2.4. A topological space X is *sober*, if any irreducible closed subset has a unique generic point. X is *quasi-separated* ([EGA4], Proposition 1.2.7), if the intersection of any two quasi-compact open subset of X is again quasi-compact. X is *coherent*, if it is sober, quasi-compact, quasi-separated, and admits a quasi-compact open basis.

Definition 2.5. A poset (L, \leq) is a *distributive lattice*, if

- (a) any two elements of a, b of L admit a *join* $a \vee b$, namely the supremum of $\{a, b\}$, and a *meet* $a \wedge b$, namely the infimum of $\{a, b\}$,
- (b) L has a unique maximal (resp. minimal) element 1 (resp. 0), and
- (c) distribution law holds: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

A distributive lattice L is *complete*, if the supremum is defined for any subset of L , and infinite distribution law holds.

In fact, a distributive lattice can be regarded as a semiring: recall that, an element a of a monoid M is *absorbing*, if $ax = a$ for any $x \in M$.

Definition 2.6. A 5-uple $R = (R, +, \times, 0, 1)$ is an (*idempotent*) *semiring* if

- (a) R is a set, $+$, \times are two binary operators on R , and $0, 1$ are two elements of R ,

- (b) $(R, +, 0)$ is a commutative idempotent monoid, and $(R, \times, 1)$ is a commutative monoid,
- (c) 0 (resp. 1) is an absorbing element with respect to the multiplication (resp. the addition), and
- (d) distribution law holds.

Let R be a semiring. Then, R can be regarded as a poset by setting

$$a \leq b \Leftrightarrow a + b = b.$$

Then $a + b$ gives the supremum of a, b with respect to this order, and $0, 1$ become the maximum and the minimum of R , respectively. In fact,

Proposition 2.7. To give a distributive lattice is equivalent to giving a semiring with idempotent multiplication.

Indeed, if we are given a semiring R with idempotent multiplication, then we may replace $+$ by \vee and \times by \wedge .

In particular, a distributive lattice can be regarded as an algebra. Let (\mathbf{DLat}) (resp. (\mathbf{CDLat})) be the category of distributive lattices (resp. complete distributive lattices) and their homomorphisms. These categories are finitary algebraic categories. As a consequence, (\mathbf{DLat}) is small complete and small co-complete.

The underlying functor $U : (\mathbf{CDLat}) \rightarrow (\mathbf{DLat})$ admits a left adjoint $\langle \underline{\text{comp}} \rangle$, defined as follows: for a distributive lattice L , $\langle \underline{\text{comp}} \rangle(L)$ is the set of all non-empty *lower sets* of L :

$$\langle \underline{\text{comp}} \rangle(L) = \{\emptyset \neq S \subset L \mid x \leq y, y \in S \Rightarrow x \in S\}.$$

$S \vee T$ (resp. $S \wedge T$) is defined as the lower set generated by $s \vee t$ (resp. $s \wedge t$) for $s \in S, t \in T$ respectively.

Let $\mathbf{1}$ be the initial object in (\mathbf{DLat}) . This is the simplest Boolean lattice $\{0, 1\}$.

Definition 2.8. (1) For a topological space X , $\Omega(X)$ is the set of open subsets of X . This becomes a complete distributive lattice via setting $\vee = \cup$ and $\wedge = \cap$, and a continuous map $f : X \rightarrow Y$ between two topological spaces induces a lattice homomorphism $f^{-1} : \Omega(Y) \rightarrow \Omega(X)$. Therefore, we have a contravariant functor $\Omega : (\mathbf{Top})^{\text{op}} \rightarrow (\mathbf{CDLat})$.

- (2) Conversely, for a complete distributive lattice L , let $\langle \underline{\text{pt}} \rangle(L)$ be the set of homomorphisms $L \rightarrow \mathbf{1}$. This has a natural topology, the open set of which is of the form

$$\phi(a) = \{p \in \langle \underline{\text{pt}} \rangle(L) \mid p(a) = 1\}$$

where $a \in L$. Then $\langle \underline{\text{pt}} \rangle(L)$ becomes a sober space. If we denote the category of sober spaces by (\mathbf{Sob}) , then we have a contravariant functor $\langle \underline{\text{pt}} \rangle : (\mathbf{CDLat}) \rightarrow (\mathbf{Sob})^{\text{op}}$.

The functor Ω is not essentially surjective. The objects in the image category of Ω is called *spatial*, and we denote by (\mathbf{SCDLat}) the full subcategory of (\mathbf{CDLat}) consisting of spatial complete distributive lattices. It turns out that Ω and $\langle \underline{\text{pt}} \rangle$ gives an equivalence of categories $(\mathbf{Sob})^{\text{op}} \simeq (\mathbf{SCDLat})$. This also gives the right adjoint of the underlying functor $(\mathbf{Sob}) \rightarrow (\mathbf{Top})$, namely the *soberification* $\langle \underline{\text{pt}} \rangle \Omega$, and the left adjoint of $(\mathbf{SCDLat}) \rightarrow (\mathbf{CDLat})$. Summarizing, we have a commutative diagram of adjunctions:

$$\begin{array}{ccc} (\mathbf{Top})^{\text{op}} & \xrightleftharpoons[U]{\quad} & (\mathbf{Sob})^{\text{op}} \\ \Omega \downarrow & & \uparrow \Omega \simeq \\ (\mathbf{CDLat}) & \xrightleftharpoons[U]{\quad} & (\mathbf{SCDLat}) \end{array}$$

Let us see for distributive lattices. The correspondence between the spaces and lattices become more clear in this case.

For a distributive lattice L , $\langle \underline{\text{comp}} \rangle(L)$ turns out to *have enough points*, namely, $\langle \underline{\text{comp}} \rangle(L)$ is spatial. This is a consequence of a more general statement:

Proposition 2.9. Let R be a semiring, and I an ideal of R which is not the unit ideal. Then, there exists a maximal ideal containing I .

Actually, we use this proposition when we prove the existence of a maximal ideal of a given ring.

When R is a distributive lattice, the ideals of R correspond to the lower sets of R .

Theorem 2.10 (Stone duality). (1) For a distributive lattice L , $\langle \underline{\text{pt}} \rangle \langle \underline{\text{comp}} \rangle(L)$ becomes a coherent space, and a homomorphism $L \rightarrow M$ induces a quasi-compact morphism $\langle \underline{\text{pt}} \rangle \langle \underline{\text{comp}} \rangle(M) \rightarrow \langle \underline{\text{pt}} \rangle \langle \underline{\text{comp}} \rangle(L)$. Hence, we have a functor $\langle \underline{\text{pt}} \rangle \langle \underline{\text{comp}} \rangle : (\mathbf{DLat}) \rightarrow (\mathbf{Coh})^{\text{op}}$.

- (2) Conversely, for a coherent space X , let $\Omega_c(X)$ be the set of quasi-compact open subsets of X . Then, $\Omega_c(X)$ becomes a distributive lattice and Ω_c induces a functor $(\mathbf{Coh})^{\text{op}} \rightarrow (\mathbf{DLat})$. This gives the inverse of $\langle \text{pt} \rangle \langle \text{comp} \rangle$, and hence an equivalence of categories.

The functor $\langle \text{comp} \rangle$ corresponds to the underlying functor $(\mathbf{Coh}) \rightarrow (\mathbf{Sob})$: note that this is *not* fully faithful, as we only consider quasi-compact morphisms between coherent spaces.

Summarizing, we have a commutative square:

$$\begin{array}{ccc} (\mathbf{Coh})^{\text{op}} & \xrightleftharpoons{U} & (\mathbf{Sob})^{\text{op}} \\ \simeq \updownarrow \Omega_c & & \simeq \updownarrow \Omega \\ (\mathbf{DLat}) & \xrightleftharpoons{U} & (\mathbf{SCDLat}) \end{array}$$

Since (\mathbf{DLat}) is small complete and co-complete, so is (\mathbf{Coh}) . This justifies our standing point that we stick on to coherent spaces.

Remark 2.11. Here, we will explain some other aspects, which give reasonings of sticking to coherent spaces.

- (1) Let $R = k^{\prod X}$ be a direct product of the copies of a field k , with an infinite index set X . Note that $\text{Spec } R \neq \coprod_X \text{Spec } k = X$, since $\text{Spec } R$ is quasi-compact, while $X = \coprod_X \text{Spec } k$ is not: X is discrete, and $\text{Spec } R = \beta X$, the Stone-Ćech compactification of X . βX is a Hausdorff coherent space, in other words, a totally disconnected compact Hausdorff space. A point x on $\beta X \setminus X$ is a prime ideal corresponding to a non-principal ultrafilter on X . The stalk $\mathcal{O}_{X,x}$ is a non-standard extension field of k , if the cardinality of k is not less than $\#X$. (See for example, [CN] or other textbooks on set theory or model theory.)

This is saying that considering coproducts of the underlying space in (\mathbf{Top}) might not be the best choice, since product of rings and coproduct of schemes does not coincide. In contrast, when we consider coproducts in (\mathbf{Coh}) , then $\coprod_X \text{Spec } k$ coincides with $\text{Spec } R$. This implies that we are ignoring some data of the algebra, when we regard the underlying space merely as a topological space, especially when we consider infinite operations.

- (2) One of the motivation of extending the notion of schemes is from arithmetics: we want to handle the infinite places of a given number field equally with the finite places. However, the archimedean-complete field \mathbb{R} appears only after some transcendental operations. Let R be a subring of $\mathbb{Q}^{\mathbb{N}}$ whose element $(a_n)_{n \in \mathbb{N}}$ is uniformly bounded: namely, there is an upper bound $M > 0$ such that $|a_n| < M$ for any n . Then, \mathbb{R} appears as the residue field R/\mathfrak{m} of the maximal ideal \mathfrak{m} of R , corresponding to a non-principal ultrafilter \mathcal{U} on \mathbb{N} :

$$\mathfrak{m} = \{(a_n)_n \mid \forall \epsilon > 0, |a_n| < \epsilon \text{ a.e. } \mathcal{U}\}.$$

This hints us that, these transcendental points may bridge the gap between the archimedean world and the non-archimedean world.

Before we go on to the next section, we mention a simple, but important fact on sheaves on coherent spaces. Let X be a topological space. Since $\Omega(X)$ and $\Omega_c(X)$ are posets, they can be regarded as a category: the object set is $\Omega(X)$ (resp. $\Omega_c(X)$) and there is a unique morphism $a \rightarrow b$ if and only if $a \leq b$.

Proposition 2.12. Let X be a coherent space, and \mathcal{C} be a small complete category. Then, to give a \mathcal{C}^{op} -valued sheaf on X is equivalent to give a finite continuous functor $\Omega_c(X) \rightarrow \mathcal{C}$.

Proof. Let $F : \Omega_c(X) \rightarrow \mathcal{C}$ be a finite continuous functor. We will define a continuous functor $\tilde{F} : \Omega(X) \rightarrow \mathcal{C}$ as follows: any $U \in \Omega(X)$ is a union of quasi-compact subsets U_i 's of X . Then, $\tilde{F}(U)$ is defined as the equalizer of $\prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \cap U_j)$. This does not depend on the choice of the refinement $\{U_i\}_i$. \square

3 Universality of coherent schemes

In this section, we characterize the category of coherent schemes by a universal property. Note that this cannot be achieved only by specifying the algebraic type; we must also fix the Grothendieck topology.

The essential idea of the construction of general schemes is already given in [TV].

3.1 The definition of \mathcal{C} -schemes

In the sequel, we fix the following data: \mathcal{C} is a category which admits pull backs and finite coproducts.

Definition 3.1. Let \mathcal{E} be a lluf subcategory of \mathcal{C} (namely, $\mathcal{E} \rightarrow \mathcal{C}$ is faithful and essentially surjective) such that

- (a) \mathcal{E} is closed under isomorphisms, and base change: namely, if $a \rightarrow b$ is a morphism in \mathcal{E} , then $a \times_b c \rightarrow c$ is also in \mathcal{E} .
- (b) \mathcal{E} admits pushouts, and $\mathcal{E} \rightarrow \mathcal{C}$ is pushout preserving. In particular, the initial object of \mathcal{C} coincides with that of \mathcal{E} .
- (c) Any morphism $f : A \rightarrow R$ in \mathcal{E} is monic and *flat*, namely $A \times_R (-)$ is coproduct preserving.
- (d) If $\{U_i \rightarrow X\}_i$ is a finite set of \mathcal{E} -morphisms, then the diagram

$$\coprod_{i,j} U_i \times_X U_j \xrightarrow{p_1, p_2} \coprod_i U_i$$

has a coequalizer in \mathcal{E} , where $p_1 : U_i \times_X U_j \rightarrow U_i$ and $p_2 : U_i \times_X U_j \rightarrow U_j$ are canonical morphisms.

Definition 3.2. $\mathcal{O} = \{\mathcal{O}_X\}_{X \in \mathcal{C}}$ is a family indexed by the objects in \mathcal{C} , and \mathcal{O}_X is a family of finite sets $\{U_i \rightarrow X\}_i^{<\infty}$ of morphisms in \mathcal{E} , which satisfies the following condition:

- (1) the canonical fork

$$\coprod_{i,j} U_i \times_X U_j \xrightarrow{p_1, p_2} \coprod_i U_i \rightarrow X$$

is a coequalizer. This is what we call *descent datum*.

- (2) \mathcal{O} gives a Grothendieck topology on \mathcal{C} , namely:

- (a) $\{X \rightarrow X\}$ is in \mathcal{O}_X .
- (b) If $S_1 \subset S_2$ is an inclusion of finite set of \mathcal{E} -morphisms over $X \in \mathcal{C}$ and $S_1 \in \mathcal{O}_X$, then $S_2 \in \mathcal{O}_X$.
- (c) If $X \rightarrow Y$ is a morphism in \mathcal{C} , and $S \in \mathcal{O}_Y$, then $X \times_Y S \in \mathcal{O}_X$, where

$$X \times_Y S = \{X \times_Y U \rightarrow X \mid [U \rightarrow Y] \in S\}.$$

- (d) If $S \in \mathcal{O}_X$ and $T = \{U_i \rightarrow X\}_i$ are finite sets of \mathcal{E} -morphisms such that $V \times_X T \in \mathcal{O}_V$ for any $[V \rightarrow X] \in S$, then $T \in \mathcal{O}_X$.

Example 3.3. Here, we will give some examples of coherent topological categories.

- (1) The category **(Coh)** of coherent spaces, or equivalently, the opposite category $(\mathbf{DLat})^{\text{op}}$ of distributive lattices: \mathcal{E} is the subcategory of open immersions, and $\{U_i \rightarrow X\} \in \mathcal{O}$ if and only if $\{U_i\}$ covers X , in the usual sense.
- (2) The opposite category $(\mathbf{CRing})^{\text{op}}$ of commutative ring with units: \mathcal{E} is the subcategory of localizations $S^{-1}R \rightarrow R$ of finite type, namely the multiplicative system S is finitely generated (hence $S^{-1}R \simeq R_f$ for some $f \in R$) $\{R_{f_i} \rightarrow R\}_i \in \mathcal{O}_R$ if and only if $(f_i)_i$ generates the unit ideal. We will discuss the generalization of this example in section §5.
- (3) The opposite category $(\mathbf{CMnd}_\infty)^{\text{op}}$ of commutative monoids with an absorbing element. \mathcal{E} is the subcategory of localizations of finite type, similar to the case of rings. $\{R_{f_i} \rightarrow R\}$ is an element of \mathcal{O}_R if and only if one of the f_i 's is a unit.

Definition 3.4. (1) We will refer to the triple $\mathcal{C} = (\mathcal{C}, \mathcal{E}, \mathcal{O})$ as the *coherent topological category*.

- (2) Let $\mathcal{C}_i = (\mathcal{C}_i, \mathcal{E}_i, \mathcal{O}^i)$ ($i = 1, 2$) be two coherent topological categories. A morphism of coherent topological categories $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a functor such that $F(\mathcal{E}_1) \subset \mathcal{E}_2$ and $F(\mathcal{O}^1) \subset \mathcal{O}^2$.
- (3) We denote by **(CT-Cat)** the 2-category of coherent topological categories.

The crucial difference between the category of rings and that of schemes is that we can “patch” objects along open subobjects. Here, we will axiomatize what “patching” means.

Definition 3.5. Let $\mathcal{C} = (\mathcal{C}, \mathcal{E}, \mathcal{O})$ be a coherent topological category. \mathcal{C} is *schematic*, if any cocartesian diagram

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & b \amalg_a c \end{array}$$

in \mathcal{C} is bicartesian, namely a is the pullback of $[b \rightarrow b \amalg_a c \leftarrow c]$.

We will denote by **(CT-Sch)** the full subcategory of **(CT-Cat)** consisting of schematic coherent topological categories. The main theorem of this paper is as follows:

Theorem 3.6. The underlying functor $U : (\mathbf{CT-Sch}) \rightarrow (\mathbf{CT-Cat})$ admits a left adjoint $\langle \underline{\text{Sch}} \rangle$.

The next section is devoted to the proof.

Remark 3.7. Note that until now, we haven't assumed completeness of the category \mathcal{C} . This enables us to apply $\langle \underline{\text{Sch}} \rangle$ to the category of schemes, and we can say that the above adjoint is idempotent. Note that the category of ordinary schemes is *not* complete, for example, we cannot define an infinite product of the projective space $\prod^\infty \mathbb{P}^1$. The reason of this incompleteness relies on the fact that we assume schemes to be locally isomorphic to the spectrum of a ring: $\prod^\infty \mathbb{P}^1$ has the product topology, which means that there are no open affine subsets. This is another motivation of considering weak \mathcal{C} -schemes later on.

3.2 Proof of Theorem 3.6

To begin with, we must construct the spectrum $\text{Spec}^0 R$ for each object R of a coherent topological category \mathcal{C} .

Definition 3.8. Let \mathcal{C} be a coherent topological category, and $R \in \mathcal{C}$.

- (1) $\Omega_0(R)$ is the family of finite sets $\{U_i \rightarrow R\}_i$ of \mathcal{C} -morphisms.
- (2) We define a relation \prec on $\Omega_0(R)$ by

$$\{U_i\}_i \prec \{V_j\}_j \Leftrightarrow \{U_i \times_R V_j \rightarrow U_i\}_j \in \mathcal{O}_{U_i} \ (\forall i).$$

This relation satisfies the reflexivity and transitivity.

- (3) Let \equiv be the equivalence relation generated by \prec , namely

$$a \equiv b \Leftrightarrow a \prec b \text{ and } b \prec a.$$

Set $\Omega_1(R) = \Omega_0(R) / \equiv$.

(4) We can define \vee and \wedge on $\Omega_1(R)$ by

$$\begin{aligned}\{U_i\}_{i \in I} \vee \{U_i\}_{i \in J} &= \{U_i\}_{i \in I \amalg J}, \\ \{U_i\}_i \wedge \{V_j\}_j &= \{U_i \times_R V_j\}_{ij}\end{aligned}$$

This does not depend on the representation, and $\Omega_1(R)$ becomes a distributive lattice: 0 is an empty set, and 1 is $\{R \rightarrow R\}$.

(5) Let $A \rightarrow R$ be a morphism in \mathcal{C} . Then, we have a morphism $\Omega_1(R) \rightarrow \Omega_1(A)$ defined by $\{U_i\}_i \mapsto \{A \times_R U_i\}_i$. This gives a contravariant functor $\Omega_1 : \mathcal{C}^{\text{op}} \rightarrow (\mathbf{DLat})$. We will denote the functor $\langle \text{pt} \rangle \langle \text{comp} \rangle \Omega_c(-) : \mathcal{C} \rightarrow (\mathbf{Coh})$ by Spec^0 .

Proof. Here, we will only give a brief proof of (2) and (4). In the sequel, we omit the subscript of \times_R and simply write \times .

(2) We first show the reflexivity: $\{U_i\}_i \prec \{U_i\}_i$. For any j , $U_j \times U_j$ is isomorphic to U_j , since $U_j \rightarrow R$ is monic. Hence, $\{U_j \rightarrow U_j\} \in \mathcal{O}_{U_j}$. From the condition (b) of Definition 3.2, we have that $\{U_i \times U_j \rightarrow U_j\}_j \in \mathcal{O}_{U_j}$.

Next, we show the transitivity: suppose $\{U_i\}_i \prec \{V_j\}_j$ and $\{V_j\}_j \prec \{W_k\}_k$. Then, $\{W_k \times V_j \rightarrow V_j\}_k \in \mathcal{O}_{V_j}$ for any j , and (c) shows that $\{W_k \times V_j \times U_i \rightarrow V_j \times U_i\}_k \in \mathcal{O}_{U_i \times V_j}$ for any i, j . Combining with the fact that $\{U_i \times V_j \rightarrow U_i\}_j \in \mathcal{O}_{U_i}$ for any i , (d) tells that $\{U_i\}_i \prec \{W_k\}_k$.

(4) We will only prove for the join. Suppose $\{U_i\}_i \prec \{\tilde{U}_k\}_k$ and $\{V_j\}_j \prec \{\tilde{V}_l\}_l$. Then, $\{V_j \times \tilde{V}_l \rightarrow V_j\}_l \in \mathcal{O}_{V_j}$ tells that

$$\{U_i \times \tilde{U}_k \times V_j \times \tilde{V}_l \rightarrow U_i \times \tilde{U}_k \times V_j\}_l \in \mathcal{O}_{U_i \times \tilde{U}_k \times V_j}.$$

Also, $\{\tilde{U}_k \times U_i \rightarrow U_i\}_k \in \mathcal{O}_{U_i}$ implies

$$\{U_i \times \tilde{U}_k \times V_j \rightarrow U_i \times V_j\}_k \in \mathcal{O}_{U_i \times V_j}.$$

Combining these two, we obtain $\{U_i \times V_j\}_{ij} \prec \{\tilde{U}_k \times \tilde{V}_l\}_{kl}$.

□

Definition 3.9. (1) On a coherent space X , we have a canonical (\mathbf{DLat}) -valued sheaf $\tau_X : \Omega_c(X) \rightarrow (\mathbf{DLat})^{\text{op}}$ defined by $U \mapsto \Omega_c(U)$. A morphism $f : X \rightarrow Y$ of coherent spaces induces a morphism $f^{-1} : \tau_Y \rightarrow f_* \tau_X$ of (\mathbf{DLat}) -valued sheaves on X , defined by

$$\Omega_c(U) = \tau_Y(U) \rightarrow \tau_X(f^{-1}U) = \Omega_c(f^{-1}U) \quad (V \mapsto f^{-1}V).$$

- (2) On a \mathcal{C}^{op} -valued coherent space (X, \mathcal{O}_X) , we have a canonical (\mathbf{DLat}) -valued sheaf $\sigma_X : \Omega_c(X) \rightarrow (\mathbf{DLat})^{\text{op}}$ defined by the sheafification of $U \mapsto \Omega_1(\mathcal{O}_X(U))$.
- (3) A *weak \mathcal{C} -scheme* is a triple $X = (X, \mathcal{O}_X, \beta_X)$, where (X, \mathcal{O}_X) is a \mathcal{C}^{op} -coherent space, and $\beta_X : \sigma_X \rightarrow \tau_X$ is a morphism of $(\mathbf{DLat})^{\text{op}}$ -valued sheaves on X , which satisfies the following condition:

for any inclusion $V \subset U$ of quasi-compact open subsets of X , the restriction functor $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$ factors through the \mathcal{C} -morphism $j : A \rightarrow \mathcal{O}_X(U)$ whenever $\beta_X(U)(\{A \rightarrow \mathcal{O}_X(U)\}) \geq V$.

We will refer to β_X as the “support morphism” on X : we will explain the reason of this name later in 4.22.

Also, we will denote the underlying coherent space by $|X|$ to avoid confusion.

- (4) A morphism $f : X \rightarrow Y$ of weak \mathcal{C} -schemes is a pair $(f, f^\#)$, where $f : |X| \rightarrow |Y|$ is a quasi-compact morphism, and $f^\# : f_*\mathcal{O}_X \rightarrow \mathcal{O}_Y$ is a morphism of \mathcal{C}^{op} -valued sheaves on Y which makes the following diagram commutative:

$$\begin{array}{ccc} \sigma_Y & \xrightarrow{f^\#} & f_*\sigma_X \\ \beta_Y \downarrow & & \downarrow f_*\beta_X \\ \tau_Y & \xrightarrow{f^{-1}} & f_*\tau_X \end{array} \quad ,$$

where the top arrow is defined by

$$\sigma_Y(U) = \Omega_1(\mathcal{O}_Y(U)) \xrightarrow{\Omega_1 f^\#} \Omega_1(\mathcal{O}_X(f^{-1}U)) = \sigma_X(f^{-1}U).$$

- (5) We denote by **(w. \mathcal{C} -Sch)** the category of weak \mathcal{C} -schemes and their morphisms.

One advantage of this new definition is that, we can obtain the spectrum functor as the adjoint of the global section functor, as below.

Definition 3.10. The global section functor $\Gamma : (\mathbf{w.\mathcal{C}\text{-Sch}}) \rightarrow \mathcal{C}$ admits a right adjoint, namely the spectrum functor Spec . Moreover, Spec is fully faithful.

Proof. For an object $R \in \mathcal{C}$, we define a weak \mathcal{C} -scheme $X = (|X|, \mathcal{O}_X, \beta_X)$ as follows:

- (a) $|X| = \text{Spec}^0 R$,
- (b) \mathcal{O}_X is a \mathcal{C}^{op} -valued sheaf on X , in other words, a finite continuous functor $\Omega_1(R) \rightarrow \mathcal{C}$. This is defined by

$$U = \{U_i\}_i \mapsto \text{coker}(\Pi_{i,j}(U_i \times_R U_j) \xrightarrow{p_1, p_2} \Pi_i U_i).$$

This does not depend on the representation of the open set U . For an inclusion $U = \{U_i\}_i \prec \{V_j\}_j = V$ of quasi-compact open subsets of $|X|$, we have a commutative diagram

$$\begin{array}{ccccc} \Pi_{k,l}(V_k \times V_l \times U_i) & \rightrightarrows & \Pi_k(V_k \times U_i) & \longrightarrow & U_i \\ \downarrow & & \downarrow & & \downarrow \text{dotted} \\ \Pi_{k,l}(V_k \times V_l) & \rightrightarrows & \Pi_k V_k & \longrightarrow & \mathcal{O}_X(V) \end{array},$$

which patches up to give $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$.

This is the *structure sheaf* of X .

- (c) The support morphism $\beta_X : \sigma_X \rightarrow \tau_X$ is defined canonically: any quasi-compact open subset U of $|X|$ is represented by a family of \mathcal{C} -morphisms $\{U_i \rightarrow R\}$. Set $A = \mathcal{O}_X(U)$. Then, for any \mathcal{C} -morphism $V \rightarrow A$, the composition

$$U_i \times_A V \rightarrow U_i \rightarrow U$$

is an \mathcal{C} -morphism. The family of \mathcal{C} -morphisms $\{U_i \times_A V \rightarrow U\}$ defines a quasi-compact open subset $\beta_X(U)(V)$ of U , and is independent of the choice of the representation of U .

Let $A \rightarrow B$ be a morphism in \mathcal{C} . We can define a morphism

$$X = \text{Spec } A \rightarrow \text{Spec } B = Y$$

as follows: we already have the morphism between the underlying spaces $f : |X| \rightarrow |Y|$. The morphism between the structure sheaves $f^\# : f_* \mathcal{O}_X \rightarrow \mathcal{O}_Y$

is given as follows: for a quasi-compact open set $U = \{U_i\}$ of Y , we have a commutative diagram

$$\begin{array}{ccccc}
\Pi_{i,j}(A \times_B U_i \times_B U_j) & \rightrightarrows & \Pi_i(A \times_B U_i) & \longrightarrow & \mathcal{O}_X(f^{-1}U) \\
\downarrow & & \downarrow & & \downarrow \text{dotted} \\
\Pi_{i,j}(U_i \times_B U_j) & \rightrightarrows & \Pi_i U_i & \longrightarrow & \mathcal{O}_Y(U)
\end{array}$$

which gives a morphism $\mathcal{O}_X(f^{-1}U) \rightarrow \mathcal{O}_Y(U)$. We can easily check that $(f, f^\#)$ is indeed a morphism of weak \mathcal{C} -schemes.

Hence, we have a functor $\text{Spec} : \mathcal{C} \rightarrow (\mathbf{w.} \mathcal{C}\text{-}\mathbf{sch})$. The unit functor $\epsilon_X : X \rightarrow \text{Spec} \Gamma(X, \mathcal{O}_X) = \tilde{X}$ is given as follows: the support morphism $\beta_X(X)$ gives a lattice homomorphism $\Omega_1(\Gamma(X, \mathcal{O}_X)) \rightarrow \Omega_c(X)$, which induces the morphism $\epsilon : |X| \rightarrow \text{Spec}^0 \Gamma(X, \mathcal{O}_X)$ between the underlying spaces. The morphism between the structure sheaves $\epsilon_X^\# : \epsilon_* \mathcal{O}_X \rightarrow \mathcal{O}_{\tilde{X}}$ is given as follows: for each \mathcal{C} -morphism $U_i \rightarrow \Gamma(X, \mathcal{O}_X)$, there is a morphism $\mathcal{O}_X(\epsilon^{-1}U_i) \rightarrow \mathcal{O}_{\tilde{X}}(U_i)$. These patch up to give a morphism $\epsilon_* \mathcal{O}_X \rightarrow \mathcal{O}_{\tilde{X}}$.

The counit functor is the canonical morphism $\eta_R : R \rightarrow \Gamma(\text{Spec}^0 R, \mathcal{O}_{\text{Spec} R})$. Since η is a natural isomorphism, Spec is fully faithful. \square

Definition 3.11. (1) A weak \mathcal{C} -scheme which is isomorphic to a spectrum of some object in \mathcal{C} is called *affine*.

(2) A \mathcal{C} -scheme is a weak \mathcal{C} -scheme which is locally affine. Let $(\mathcal{C}\text{-}\mathbf{Sch})$ be the full subcategory of $(\mathbf{w.}\mathcal{C}\text{-}\mathbf{Sch})$ consisting of \mathcal{C} -schemes.

(3) We have a pullback in $(\mathcal{C}\text{-}\mathbf{Sch})$ constructed as follows: let X, Y be a \mathcal{C} -schemes over a \mathcal{C} -scheme S .

- (i) Case X, Y, S are all affine, say $X = \text{Spec } A$, $Y = \text{Spec } B$, $S = \text{Spec } R$ respectively. Then $X \times_S Y$ is defined as $\text{Spec}(A \times_R B)$.
- (ii) Case S is affine: let $X = \cup_i X_i$ and $Y = \cup_j Y_j$ be open affine coverings of X and Y . Then, $\{X_i \times_S Y_j\}_{ij}$ patches up to define $X \times_S Y$.
- (iii) Case S is arbitrary: Let $S = \cup_l S_l$ be an open affine covering of S , and set $X_l = \pi_X^{-1}(S_l)$, $Y_l = \pi_Y^{-1}(S_l)$ where $\pi_X : X \rightarrow S$ and $\pi_Y : Y \rightarrow S$ are structure maps. Then, $\{X_l \times_{S_l} Y_l\}_l$ patches up to give $X \times_S Y$.

These definitions does not depend on the choice of affine open covers of X, Y and S .

- (4) The lluf subcategory \mathcal{E}' of $(\mathcal{C}\text{-}\mathbf{Sch})$ are open immersions, and \mathcal{O}' is the family of finite quasi-compact open covers. Then, $((\mathcal{C}\text{-}\mathbf{Sch}), \mathcal{E}', \mathcal{O}')$ becomes a schematic coherent topological category, and the spectrum functor $\mathrm{Spec} : \mathcal{C} \rightarrow (\mathcal{C}\text{-}\mathbf{Sch})$ becomes a morphism of coherent topological categories.
- (5) Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a morphism of coherent topological categories. Then, we can extend F to a functor $(\mathcal{C}\text{-}\mathbf{Sch}) \rightarrow (\mathcal{D}\text{-}\mathbf{Sch})$ as follows:
 - (i) If $X = \mathrm{Spec} A$ is an affine \mathcal{C} -scheme, then $F(X) = \mathrm{Spec} F(A)$.
 - (ii) For a general \mathcal{C} -scheme X , let $X = \cup_i U_i$ be an affine open covering of X . Then $F(X)$ is defined by the patching of $F(U_i)$. This definition does not depend on the choice of the affine covering $\{U_i\}$.
 - (iii) Let $f : X \rightarrow Y$ be a morphism of \mathcal{C} -schemes, and $Y = \cup_i Y_i$ be an affine open covering of Y . Then, $X_i = f^{-1}(Y_i) \rightarrow Y_i$ is determined by a morphism $\Gamma(X_i, \mathcal{O}_X) \rightarrow B_i = \Gamma(Y_i, \mathcal{O}_Y)$. This induces a morphism $\Gamma(F(X_i), \mathcal{O}_{F(X)}) \rightarrow F(B_i) = \Gamma(F(Y_i), \mathcal{O}_{F(Y)})$, and these patch up to give $F(f) : F(X) \rightarrow F(Y)$.
 - (iv) In other words, we have a functor $\langle \underline{\mathbf{Sch}} \rangle : (\mathbf{CT}\text{-}\mathbf{Cat}) \rightarrow (\mathbf{CT}\text{-}\mathbf{Sch})$, sending \mathcal{C} to $(\mathcal{C}\text{-}\mathbf{Sch})$.

Remark 3.12. (1) This definition of schemes is seemingly different from the conventional one: we usually use the notion of locally ringed spaces and local homomorphisms, and in fact, the above definition is almost equivalent to this conventional one. However, the usual definition of schemes bothers us since it uses the limit process to look at the stalks. Note that we cannot take arbitrary limits in the categories of usual schemes.

- (2) There is another fancier way of defining \mathcal{C} -schemes. \mathcal{C} becomes a site by the fixed Grothendieck topology, and \mathcal{C} -scheme is a set-valued sheaf on \mathcal{C} which locally isomorphic to

$$\mathcal{C}(-, R) : A \mapsto \mathcal{C}(A, R)$$

for some $R \in \mathcal{C}$. When we consider groupoid-valued sheaves, then we can immediately define \mathcal{C} -stacks ([LMB]).

- (3) The notion of weak \mathcal{C} -schemes coincides with \mathcal{A} -schemes introduced in [T2], when \mathcal{C} is the opposite category of rings, monoids, etc. We will summarize the advantage of considering \mathcal{A} -schemes later on.

Proposition 3.13. If \mathcal{C} is schematic, then the functor $\text{Spec} : \mathcal{C} \rightarrow (\mathcal{C}\text{-Sch})$ is an equivalence of categories.

Proof. It suffices to show that any \mathcal{C} -scheme X is affine. Let $X = \cup_i^n U_i$ be an affine open covering of X . It suffices to show for $n = 2$, since the rest follows from the induction. Let $U_1 \cap U_2 = \cup_j^m V_j$ be an affine open covering of $U_1 \cap U_2$. Again, it suffices to show for $m = 1$.

Therefore, the proof is reduced to the following Lemma: \square

Lemma 3.14. Suppose the following diagram is a bicartesian square in \mathcal{C} of \mathcal{C} :

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ v \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

Then, applying the spectrum functor to this square gives a bicartesian square in $(\mathcal{C}\text{-Sch})$.

Proof. Suppose we are given a commutative diagram

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow f \\ \text{Spec } C & \xrightarrow{g} & X \end{array}$$

in $(\mathcal{C}\text{-Sch})$. We can define the unique morphism $h : \text{Spec } D \rightarrow X$ by an explicit construction:

$$h : \Omega_c(X) \rightarrow \Omega_1(D) = \Omega_1(B) \times_{\Omega_1(A)} \Omega_1(C)$$

is given by $U \mapsto (f^{-1}U, g^{-1}U)$. The morphism $h_* \mathcal{O}_{\text{Spec } D} \rightarrow \mathcal{O}_X$ is given by

$$\mathcal{O}_{\text{Spec } D}(h^{-1}U) = \mathcal{O}_B(f^{-1}U) \amalg_{\mathcal{O}_A((fu)^{-1}U)} \mathcal{O}_C(g^{-1}U) \rightarrow \mathcal{O}_X(U).$$

\square

We have come to the stage of proving Theorem 3.6.

Proof of Theorem 3.6. We only have to show that the functor $\langle \underline{\text{Sch}} \rangle$ is the left adjoint of $U : (\mathbf{CT-Sch}) \rightarrow (\mathbf{CT-Cat})$. The unit $\epsilon : \text{Id} \Rightarrow U\langle \underline{\text{Sch}} \rangle$ is the spectrum functor $\epsilon_{\mathcal{C}} = \text{Spec} : \mathcal{C} \rightarrow (\mathcal{C}\text{-Sch})$. The counit $\eta : \langle \underline{\text{Sch}} \rangle U \Rightarrow \text{Id}$ is defined by the inverse of Spec , since $\text{Spec} : \mathcal{D} \rightarrow (\mathcal{D}\text{-Sch})$ is an equivalence for schematic coherent topological category \mathcal{D} by Proposition 3.13. \square

4 Comparison theorems

4.1 The correspondence between the topology and ideals

In the previous sections, we obtained the distributive lattice from open immersions. However, in the usual definition we approach from the ideal description. In this section, we will explain how to obtain the Grothendieck topology from ideals.

Definition 4.1. Let $V = \langle \Omega, E \rangle$ be an algebraic type and \mathcal{C} a category of V -algebras. V has *self enhancing property*, if:

- (a) $\mathcal{C}(A, B)$ has a natural structure of V -algebras for any two objects $A, B \in \mathcal{C}$,
- (b) the composition is bilinear, namely the natural morphisms $f^* : \mathcal{C}(B, X) \rightarrow \mathcal{C}(A, X)$ and $f_* : \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B)$ induced from any homomorphism $f : A \rightarrow B$ of V -algebras are also homomorphisms.

Remark 4.2. V has self enhancing property if any two operators ϕ, ψ are commutative, namely if ϕ is m -ary and ψ is n -ary, then

$$\begin{aligned} \psi(\phi(x_{11}, \dots, x_{m1}), \dots, \phi(x_{1n}, \dots, x_{mn})) \\ = \phi(\psi(x_{11}, \dots, x_{1n}), \dots, \psi(x_{m1}, \dots, x_{mn})). \end{aligned}$$

Then, for any $A, B \in \mathcal{C}$, the action of the operators on $\mathcal{C}(A, B)$ can be defined, by the action on the value of homomorphisms:

$$\phi(f_1, \dots, f_n)(x) = \phi(f_1(x), \dots, f_n(x))$$

for any homomorphisms $f_i \in \mathcal{C}(A, B)$ and any n -ary operator ϕ .

Proposition 4.3. Let V be an algebraic type with self enhancing property, \mathcal{C} the category of V -algebras, and M a V -algebra. Then, the functor $\mathcal{C}(M, -) : \mathcal{C} \rightarrow \mathcal{C}$ admits a left adjoint $M \otimes (-)$.

Proof. For any $N \in \mathcal{C}$, let S be the set of quotient V -algebras of $\langle \text{free} \rangle(M \times N)$, where $\langle \text{free} \rangle : (\mathbf{Set}) \rightarrow \mathcal{C}$ is the free generator. Suppose we are given a homomorphism $\varphi : N \rightarrow \text{Hom}(M, X)$. Then, we have a homomorphism $\tilde{\varphi} : \langle \text{free} \rangle(M \times N) \rightarrow X$ induced by the map $M \times N \rightarrow X$, $(x, y) \mapsto \varphi(y)(x)$. Let $A \in S$ be the image of $\tilde{\varphi}$. Then we have a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & \mathcal{C}(M, X) \\ \downarrow & \nearrow \tilde{\varphi}_* & \\ \mathcal{C}(M, A) & & \end{array}$$

Since S is a small set, we can apply Freyd's adjoint functor theorem [CWM] to claim the existence of the left adjoint of $\text{Hom}(M, -)$. \square

As in the case of abelian categories, (\mathcal{C}, \otimes) becomes a closed symmetric monoidal category: the unit $\mathbf{1}$ with respect to \otimes is $\langle \text{free} \rangle(*)$, and the composition $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$ naturally extends to $\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$.

In the sequel, we fix an algebraic type V with self enhancing property, and \mathcal{C} the category of V -algebras.

Definition 4.4. (1) An object $R \in \mathcal{C}$ is a *monoid object* (resp. *commutative monoid object*) if there are morphisms $R \otimes R \rightarrow R$ and $\mathbf{1} \rightarrow R$ satisfying the monoid (resp. commutative monoid) axioms. We denote by $(\mathbf{Mnd}/\mathcal{C})$ (resp. $(\mathbf{CMnd}/\mathcal{C})$) the category of monoid (resp. commutative monoid) objects in \mathcal{C} .

(2) Let $R \in \mathcal{C}$ be a monoid object. An object $M \in \mathcal{C}$ is an *R -module* if there is an R -action $R \otimes M \rightarrow M$ with the unital law. When R is commutative, the category $(R\text{-mod})$ of R -modules shares similar properties with \mathcal{C} , namely it has self enhancing property, so that the symmetric monoidal structure \otimes_R is well defined and is closed. Also, there is a free generator $\langle \text{free} \rangle : (\mathbf{Set}) \rightarrow (R\text{-mod})$.

(3) An *ideal* of a commutative monoid object R is an R -submodule of R .

- (4) For two finitely generated ideals $\mathfrak{a}, \mathfrak{b}$ of a commutative monoid object R , the multiplication $\mathfrak{a}\mathfrak{b}$ is defined by the image of the morphism $\mathfrak{a} \otimes_R \mathfrak{b} \rightarrow R$. $\mathfrak{a} + \mathfrak{b}$ is defined by the image of the morphism $\mathfrak{a} \amalg \mathfrak{b} \rightarrow R$. These two operations make the set $I(R)$ of finitely generated ideals of R into a semiring.

We have a set-theoretic description of monoid objects: R is a monoid object in \mathcal{C} if R has a multiplicative monoid structure which satisfies the distribution laws:

$$\begin{aligned} a \cdot \phi(b_1, \dots, b_n) &= \phi(ab_1, \dots, ab_n), \\ \phi(a_1, \dots, a_n) \cdot b &= \phi(a_1b, \dots, a_nb) \end{aligned}$$

for any n -ary operator ϕ .

Note that for some V , the emptyset may become an ideal of a commutative monoid object: for example when V is null.

Example 4.5. The followings are algebraic types V with self enhancing property:

- (1) V is the null algebraic type, namely V -algebras are sets. In this case, \otimes is the cartesian product, and $(\mathbf{CMnd}/\mathcal{C})$ is the category of commutative monoids, in the usual sense.
- (2) V is the type of pointed sets. Then \otimes is the smash product, and $(\mathbf{CMnd}/\mathcal{C})$ is the category of commutative monoids with an absorbing element.
- (3) V is the type of abelian groups. Then \otimes is the usual tensor product, and $(\mathbf{CMnd}/\mathcal{C})$ is the category of commutative rings.
- (4) V is the type of commutative idempotent monoids with absorbing elements. $(\mathbf{CMnd}/\mathcal{C})$ is the category of semirings, except that we do not assume that 1 is absorbing with respect to $+$.

Definition 4.6. Let R be a V -algebra.

- (1) A *congruence* \equiv of R is an equivalence relation induced by a morphism $f : R \rightarrow A$ of V -algebras:

$$a \equiv b \Leftrightarrow f(a) = f(b).$$

- (2) We have a left adjoint $\langle \underline{\text{rad}} \rangle : (\mathbf{Semiring}) \rightarrow (\mathbf{DLat})$ of the underlying functor $(\mathbf{DLat}) \rightarrow (\mathbf{Semiring})$. The unit morphism $\epsilon : I(R) \rightarrow \langle \underline{\text{rad}} \rangle(I(R))$ induces a congruence E on $I(R)$. The *radical ideal* of R is an ideal \mathfrak{j} of R such that

$$a \in \mathfrak{j} \Leftrightarrow (a, 0) \in E.$$

Lemma 4.7. Let R be a semiring, and $a, b \in R$. Then, $a = b$ in $\langle \underline{\text{rad}} \rangle(R)$ if and only if there is an integer n such that $a^n \leq b$ and $b^n \leq a$.

Proof. Let \equiv be an equivalence relation on R defined by

$$a \equiv b \Leftrightarrow a^n \leq b, b^n \leq a \quad (n \gg 0).$$

Set $A = R / \equiv$. Then, the semiring structure of R descends to A , and A becomes a distributive lattice. Conversely, it is obvious that the canonical map $R \rightarrow \langle \underline{\text{rad}} \rangle(R)$ factors through A , hence $A \simeq \langle \underline{\text{rad}} \rangle(R)$. \square

Let $R \in (\mathbf{CMnd}/\mathcal{C})$ be a commutative monoid object, M an R -module and $S \subset R$ a multiplicative subset of R . The *localization* $S^{-1}M$ is defined by the universal property of making the actions of the elements of S invertible. We can give an explicit construction $S^{-1}M$ in the usual way: $S^{-1}M$ is the quotient set of $M \times S$ divided by the equivalence

$$(x, s) \equiv (y, t) \Leftrightarrow \exists u \in S, [utx = usy].$$

As usual, $S^{-1}R$ also becomes a commutative monoid object, and $S^{-1}M = S^{-1}R \otimes_R M$. Any R -multilinear operator ϕ on M can be extended to a $S^{-1}R$ -multilinear operator on $S^{-1}M$:

$$\phi\left(\frac{x_1}{s_1}, \dots, \frac{x_n}{s_n}\right) = \frac{1}{\prod_{i=1}^n s_i} \phi\left(\left(\prod_{j \neq 1} s_j\right) x_1, \dots, \left(\prod_{j \neq n} s_j\right) x_n\right).$$

If S is generated by a single element f , then $S^{-1}R$ is denoted by R_f .

Lemma 4.8. Let R be a commutative monoid object in \mathcal{C} , and $\mathfrak{a} = (f_1, \dots, f_n)$ be a finitely generated ideal of R . Then, the element of \mathfrak{a} can be written in a form $\phi(a_1 f_1, \dots, a_r f_{n_r})$, where ϕ is a derived operator in \mathcal{C} and $a_i \in R$. In particular, for any element $x \in I$, there exists an R -equivariant map $\psi : R^{\times n} \rightarrow R$ such that $\psi(f_1, \dots, f_n) = x$. Here, $R^{\times n}$ is a product of n -copies of R , regarded as an R -module.

This can be easily proven by using the induction on the length of derived operators and the distribution law.

Theorem 4.9. Let R be a commutative monoid object in \mathcal{C} , and suppose $\mathfrak{a} = (f_i)_i$ be a finitely generated ideal of R such that $\epsilon(\mathfrak{a}) = 1$ in $\langle \text{rad} \rangle(I(R))$. Then $\{R_{f_i}\}_i$ covers R , namely

$$R \rightarrow \prod_i R_{f_i} \xrightarrow{p_1, p_2} \prod_{i,j} R_{f_i f_j}$$

is an equalizer.

Proof. Let $u = (x_i/f_i^{n_i})_i \in \prod_i R_{f_i}$ be an element of the equalizer, namely $p_1(u) = p_2(u)$. This is equivalent to saying that $x_i/f_i^{n_i} = x_j/f_j^{n_j}$ in $R_{f_i f_j}$. We may assume that $f_j x_i = f_i x_j$ by replacing f_i 's by sufficiently large powers; this is a standard method, and can be seen in Hartshorne ([H], II 2.2). Since $(f_i)_i^N = 1$ in $I(R)$ for sufficiently large N by Lemma 4.7, we have $\psi(f_1, \dots, f_n) = 1$ for some R -equivariant map $\psi : R^{\times n} \rightarrow R$ by Lemma 4.8. Set $x = \psi(x_1, \dots, x_n) \in R$. Then,

$$f_i x = \psi(f_i x_j)_j = \psi(f_j x_i)_j = x_i \psi(f_j)_j = x_i.$$

This implies that $x = x_i/f_i$ in R_{f_i} so that u is in the image of $R \rightarrow \prod_i R_{f_i}$. \square

Lemma 4.10. Let R be a commutative monoid object in \mathcal{C} , and f_i, g_j elements of R . If $(f_i)_i = 1$ in $I(R)$ and $(f_i g_j)_j = 1$ in $I(R_{f_i})$ for any i , then $(g_j)_j = 1$ in $I(R)$.

Proof. For every i , we have a R -equivariant map $\psi_i : R_{f_i}^n \rightarrow R_{f_i}$ such that $\psi_i(f_i g_j)_j = 1$ in R_{f_i} . Multiplying by some power of f_i if necessary, we obtain a derived operator $\tilde{\psi}_i : R^n \rightarrow R$ such that $\tilde{\psi}_i(a_m g_{j_m})_m = f_i^N$ for some N and some $a_m \in R$. Since $(f_i)_i = 1$ in $I(R)$, we also have $(f_i^N)_i = 1$. Hence, there is a derived operator $\phi : R^m \rightarrow R$ such that $\phi(b_l f_i^N)_i = 1$ for some $b_l \in R$. Composing $\tilde{\psi}_i$'s and ϕ gives an R -equivariant map $\gamma : R^n \rightarrow R$ such that $\gamma(g_j)_j = 1$, hence the result. \square

Corollary 4.11. Let \mathcal{E} be the lluf subcategory of \mathcal{C}^{op} consisting of localizations of finite type. For each $R \in \mathcal{C}$, let \mathcal{O}_R be the family of sets $\{R_{f_i} \rightarrow R\}_i$ of \mathcal{E} -morphisms such that the ideal $(f_i)_i$ is unital. Then, $(\mathcal{C}^{\text{op}}, \mathcal{E}, \mathcal{O})$ satisfies the axioms of coherent topological category, and the induced spectrum functor $\text{Spec}^0 : \mathcal{C}^{\text{op}} \rightarrow (\mathbf{Coh})$ is the desired one:

$$\text{Spec}^0 R = \langle \text{pt} \rangle \langle \text{comp} \rangle \langle \text{rad} \rangle(I(R)).$$

Proof. First, we show that $(\mathcal{C}^{\text{op}}, \mathcal{E}, \mathcal{O})$ is indeed a coherent topological category. It is easy to see that $R_f \rightarrow R$ is indeed monic and flat in \mathcal{C}^{op} . The descent condition is satisfied by definition, hence we only need to verify that \mathcal{O} is a Grothendieck topology. The only non-trivial statement is condition (d) in Definition 3.2, but this is proved in Lemma 4.10.

Next, we show that the topology coincides with that obtained from ideals. Let R be a commutative monoid object of \mathcal{C} , and $U = \{R_{f_i} \rightarrow R\}_i$ a quasi-compact open set of $\text{Spec}^0(R)$. We define the corresponding radical ideal $\varphi(U) \in \langle \underline{\text{rad}} \rangle(I(R))$ to be the ideal generated by $(f_i)_i$. This does not depend on the representation of U , hence $\varphi : \Omega_1(R) \rightarrow \langle \underline{\text{rad}} \rangle(I(R))$ is well defined. Conversely, let $\mathfrak{a} = (f_i)_i \in \langle \underline{\text{rad}} \rangle(I(R))$ be a radical ideal. The quasi-compact open set $\psi(\mathfrak{a})$ is defined by $\{R_{f_i} \rightarrow R\}$. This again does not depend on the choice of the generators of \mathfrak{a} , hence we have a bijection between $\Omega_1(R)$ and $\langle \underline{\text{rad}} \rangle(I(R))$. Also, it is straightforward to see that this is a lattice isomorphism, and that it is functorial with respect to R . \square

Remark 4.12. For some algebraic category \mathcal{C} with its coherent topology, \mathcal{C} is already schematic, namely it is vacuous to consider \mathcal{C} -schemes. For the followings, \mathcal{C} is the opposite category of W -algebras (with the topology induced from ideals; see section §4.) The category \mathcal{C} becomes schematic, if any localization is surjective. The following cases are such:

- (1) W is the type of distributive lattices,
- (2) W is the type of idempotent semirings,
- (3) W is the algebraic type of von Neumann regular commutative rings: a commutative ring R is *von Neumann regular*, if any element $x \in R$ has a *weak inverse* $y \in R$, namely the unique element $y \in R$ which satisfies $x^2y = x$ and $xy^2 = y$. The Krull dimension of a von Neumann regular ring is 0, hence any localization becomes surjective. This is related to Serre's cohomological criterion of affineness: a scheme X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf \mathcal{F} and $i > 0$.

Hence, any von Neumann regular scheme is affine, since its Krull dimension is 0.

4.2 Comparison with classical schemes

In this section, we discuss local objects of a complete coherent topological category, and show that the notion of \mathcal{C} -scheme coincides with

- (1) the notion of coherent schemes, when \mathcal{C} is the opposite category of rings, and
- (2) the notion of monoid schemes, in the sense of Toën-Vaquié, or Deitmar, when \mathcal{C} is the opposite category of commutative monoids.

Definition 4.13. Let \mathcal{C} be a complete coherent topological category.

- (1) An object $A \in \mathcal{C}$ is *local*, if any covering of A is principal, namely: $U_i = A$ for some i if $\{U_i \rightarrow A\}_i$ is a covering of A . This is equivalent to saying that $\mathrm{Spec}^0(A)$ has a unique closed point.
- (2) A morphism $f : A \rightarrow B$ between two local objects $A, B \in \mathcal{C}$ is *local*, if the induced morphism $f : \mathrm{Spec}^0 A \rightarrow \mathrm{Spec}^0 B$ sends the unique closed point of $\mathrm{Spec}^0 A$ to that of $\mathrm{Spec}^0 B$. In the language of distributive lattices, this is equivalent to

$$f^{-1}(U) = 1 \Rightarrow U = 1$$

for any quasi-compact open $U \subset B$.

- (3) A \mathcal{C} -coherent space (X, \mathcal{O}_X) is *local*, if the stalk $\mathcal{O}_{X,x} = \varprojlim_{x \in U} \mathcal{O}_X(U)$ is local for any $x \in X$.
- (4) A morphism $f : X \rightarrow Y$ of $\mathcal{C}^{\mathrm{op}}$ -spaces is *local*, if the induced morphism $f : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,f(x)}$ on the stalks is local.
- (5) We denote by $(\mathcal{C}\text{-}\mathbf{LCS})$ the category of locally $\mathcal{C}^{\mathrm{op}}$ -spaces and local morphisms.

The following proposition shows that the category of \mathcal{C} -schemes coincides with the notion of coherent schemes, when $\mathcal{C}^{\mathrm{op}}$ is the opposite category of commutative rings:

Theorem 4.14. We have a natural fully faithful functor $(\mathbf{w.} \ \mathcal{C}\text{-}\mathbf{Sch}) \rightarrow (\mathcal{C}\text{-}\mathbf{LCS})$.

Proof. Let $X = (X, \mathcal{O}_X, \beta_X)$ be a weak \mathcal{C} -scheme. First, we will show that X is a local $\mathcal{C}^{\mathrm{op}}$ -space. Fix a point $x \in X$, and let $Y = \mathrm{Spec}^0 \mathcal{O}_{X,x} = \cup_i U_i$ be a quasi-compact open cover of Y . We have a natural isomorphism

$Y = \varprojlim_{x \in U} \Omega_c(\text{Spec } \mathcal{O}_X(U))$ since Spec is a right adjoint, and β_X induces a morphism

$$\iota : X_x = \varprojlim_{x \in U} U \rightarrow Y.$$

The left-hand side is a local object in **(DLat)**. Since $\{U_i\}_i$ covers Y , one of $\iota^{-1}U_i$ must coincide with X_x . Since $\Omega_1(X_x)$ is obtained by localizations of open neighborhoods of x , $\iota^{-1}U_i = V$ for some quasi-compact open neighborhood V of x . This shows that $U_i = V$ in $\Omega_1(\mathcal{O}_X(V))$, and hence also in $\Omega_1(\mathcal{O}_{X,x})$. Therefore, X is a local \mathcal{C}^{op} -space.

Suppose we have a morphism $f : X \rightarrow Y$ of weak \mathcal{C} -schemes. We need to show that the induced morphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ is local, for every $x \in X$ and $y = f(x)$. We have a commutative diagram

$$\begin{array}{ccc} \Omega_1(\mathcal{O}_{Y,y}) & \xrightarrow{f^{-1}} & \Omega_1(\mathcal{O}_{X,x}) \\ \beta_Y \downarrow & & \downarrow \beta_X \\ \Omega_c(Y_y) & \longrightarrow & \Omega_c(X_x) \end{array}$$

of distributive lattices. Suppose $U \in \Omega_1(\mathcal{O}_{Y,y})$ maps to $1 \in \Omega_1(\mathcal{O}_{X,x})$. Then, $\beta_X(f^{-1}U) = 1$: this implies that $x \in \beta_X(f^{-1}U)$, and hence $y \in \beta_Y(U)$. This shows that $U = 1$, which means that $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ is indeed a local morphism. As a consequence, we have a functor $\iota : (\mathbf{w}\mathcal{C}\text{-Sch}) \rightarrow (\mathcal{C}\text{-LCS})$.

It remains to prove that ι is fully faithful. Let X, Y be two weak \mathcal{C} -schemes, and $f : X \rightarrow Y$ a morphism of locally \mathcal{C}^{op} -spaces. For any inclusion of two quasi-compact open subsets $V \subset U$ of Y , it suffices to show that $f^{-1}(\beta_Y(U)(V)) = \beta_X(f^\#(U)(V))$. Let x be any point in $\beta_X(f^\#(U)(V))$, and set $y = f(x)$. We have a sequence of local morphisms

$$\Omega_1(\mathcal{O}_{Y,y}) \rightarrow \Omega_1(\mathcal{O}_{X,x}) \rightarrow \Omega_c(X_x),$$

which shows that $V = 1$ in $\Omega_1(\mathcal{O}_{Y,y})$. We have another sequence of local morphisms

$$\Omega_1(\mathcal{O}_{Y,y}) \rightarrow \Omega_c(Y_y) \rightarrow \Omega_c(X_x),$$

induced by $f^{-1} \circ \beta_Y$, and this sends 1 to 1 . This is equivalent to saying that $x \in f^{-1}(\beta_Y(U)(V))$. Therefore, $f^{-1}(\beta_Y(U)(V)) \supset \beta_X(f^\#(U)(V))$. The converse can be proved similarly. \square

Hence, we see that the notion of \mathcal{C} -scheme coincides with that of conventional schemes, when we set \mathcal{C} and Grothendieck topology properly.

Example 4.15. (1) Let V be the algebraic type of abelian groups. Then, the commutative monoid objects in the category of V -algebras are exactly commutative rings. For a ring R , $I(R)$ is the set of *finitely generated* ideals of R . Note that $\langle \text{comp} \rangle \langle \text{rad} \rangle (R(I))$ is the set of radical ideals, and $\langle \text{pt} \rangle \langle \text{comp} \rangle \langle \text{rad} \rangle (R(I))$ is the usual Zariski spectrum of R .

Let $\mathcal{C} = (\mathbf{CRing})^{\text{op}}$ be the opposite category of commutative unital rings, and \mathcal{E} the full subcategory of localizations of finite type. Then we see that \mathcal{C} -schemes are exactly coherent schemes.

- (2) When V is the null algebraic type, the commutative monoid objects in the category of V -algebras are exactly commutative monoids. Then, we obtain the coherent monoid schemes, in the sense of Deitmar, of equivalently, that of Toën-Vaquié [TV].

Remark 4.16. The category of coherent schemes include all noetherian schemes and affine schemes, which are practically all the schemes we treat. Moreover, the morphisms between noetherian (or affine) schemes are all quasi-compact. This tells that the category of coherent schemes is sufficiently large to work on. Also, Remark 3.7 implies that we had better work within this category.

4.3 Comparison with \mathcal{A} -schemes

In this section, we compare weak \mathcal{C} -schemes with \mathcal{A} -schemes introduced in [T1]. We recall the definition of \mathcal{A} -schemes, in the most general type.

Definition 4.17. A quadruple $\mathcal{A} = (W, \alpha_1, \alpha_2, \gamma)$ is a *schematizable algebraic type*, if

- (a) W is an algebraic type with commutative multiplicative monoid structure. We denote by $(W\text{-}\mathbf{alg})$ the category of W -algebras and their homomorphisms.
- (b) α_1 is a functor $(W\text{-}\mathbf{alg}) \rightarrow (\mathbf{DLat})$.
- (c) α_2 is a natural transformation $\text{Id}_{(W\text{-}\mathbf{alg})} \Rightarrow \alpha_1$ such that $\alpha_{2,R} : R \rightarrow \alpha_1 R$ is multiplication-preserving.
- (d) For each W -algebra R and a multiplicative system S , γ is a natural isomorphism $\alpha_1(S^{-1}R) \rightarrow (\alpha_{2,R}(S))^{-1}R$.

When \mathcal{F} is a $(W\text{-}\mathbf{alg})$ -valued sheaf on a coherent space X , then the functor α_1 induces a (\mathbf{DLat}) -valued sheaf $\alpha_1\mathcal{F}$ on X , defined by the sheafification of $U \mapsto \alpha_1\mathcal{F}(U)$.

Definition 4.18. Let $\mathcal{A} = (W, \alpha_1, \alpha_2, \gamma)$ be a schematizable algebraic type. An \mathcal{A} -scheme is a triple $(X, \mathcal{O}_X, \beta_X)$ such that

- (1) X is a coherent space, \mathcal{O}_X is a $(W\text{-}\mathbf{alg})$ -valued sheaf on X , $\beta_X : \alpha_1\mathcal{O}_X \rightarrow \tau_X$ is a morphism of (\mathbf{DLat}) -valued sheaves on X which we refer to as the “support morphism”, and
- (2) the restriction maps reflect localizations: let $V \subset U$ be an inclusion of quasi-compact open subsets of X , and we denote by $\mathcal{O}_X(U)_V$ the localization of $\mathcal{O}_X(U)$ by the multiplicative system

$$\{f \in \mathcal{O}_X(U) \mid \beta_X \alpha_2(f) \geq V\}.$$

A morphism $f : X \rightarrow Y$ is a pair $(f, f^\#)$, where

- (a) $f : |X| \rightarrow |Y|$ is a morphism of coherent spaces,
- (b) $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of $(W\text{-}\mathbf{alg})$ -valued sheaves on Y , such that the following diagram commutes:

$$\begin{array}{ccc} \alpha_1\mathcal{O}_Y & \xrightarrow{\alpha_1 f^\#} & f_*\alpha_1\mathcal{O}_X \\ \beta_Y \downarrow & & \downarrow f_*\beta_X \\ \tau_Y & \xrightarrow{f^{-1}} & f_*\tau_X \end{array}$$

Here, we list up the properties of \mathcal{A} -schemes: let $(\mathcal{A}\text{-}\mathbf{Sch})$ be the category of \mathcal{A} -schemes and their morphisms. Then:

Theorem 4.19 ([T2]). (1) $(\mathcal{A}\text{-}\mathbf{Sch})$ is a full subcategory of locally $(W\text{-}\mathbf{alg})$ -spaces.

- (2) The category of \mathcal{A} -schemes is small complete and small co-complete.
- (3) We can consider the image \mathcal{A} -scheme for any morphism of \mathcal{A} -schemes.

Practically, we only consider the following case:

Example 4.20. V is a self enhancing algebraic type, and W is the algebraic type of commutative monoid objects in V -algebras. For each W -algebra R , $\alpha_1(R)$ is the set of finitely generated ideals of R , divided by the congruence defined by

$$\mathfrak{a} \equiv \mathfrak{b} \Leftrightarrow \mathfrak{a}^n \leq \mathfrak{b}, \mathfrak{b}^n \leq \mathfrak{a} \quad (n \gg 0)$$

(cf. Lemma 4.7). $\alpha_2 : R \rightarrow \alpha_1(R)$ sends $a \in R$ to the principal ideal generated by a . This preserves multiplication. For each multiplicative system S of R , we have a natural isomorphism $\gamma : \alpha_1(S^{-1}R) \simeq \alpha_2(S)^{-1}\alpha_1(R)$.

The set $\alpha_1(R)$ can be regarded as the distributive lattice corresponding to $\text{Spec } R$, and we identify a finitely generated ideal $\mathfrak{a} \in \alpha_1(R)$ with the quasi-compact open subset which is the complement of the support of \mathfrak{a} . Note that $\alpha_1\mathcal{O}_X$ corresponds to σ_X appeared in the definition of weak \mathcal{C} -schemes (cf. Definition 3.9).

Then, it is straightforward to see that

Theorem 4.21. Let $\mathcal{A} = (W, \alpha_1, \alpha_2, \gamma)$ be as above. Then, the category of \mathcal{A} -schemes is equivalent to weak \mathcal{C} -schemes defined in section §5.

Example 4.22. (1) Suppose \mathcal{A} is the schematizable algebraic type induced from rings: namely, V is the algebraic type of abelian groups in Example 4.20. Then for each ring R , $\alpha_1(R)$ is the idempotent semiring of finitely generated ideals of R modulo the congruence generated by $\mathfrak{a}^2 = \mathfrak{a}$. For each element \mathfrak{a} , β_X simply gives the complement of the support of \mathfrak{a} which is a quasi-compact open subset of $\text{Spec } R$. This is why we call β_X ‘the support morphism’.

Actually, the data β_X is equivalent to saying that (X, \mathcal{O}_X) is a locally ringed space by Theorem 4.14.

- (2) Let \mathcal{A} be as in (1). Let U be a non-empty open subset of a coherent scheme X . The Zariski-Riemann space $\text{ZR}(U, X)$ is the limit of U -admissible blowups, hence an object of \mathcal{A} -schemes, but in general not a scheme. However, this object happens to be useful to prove pure algebro-geometric theorems, such as Nagata embeddings ([T2], [T3]).
- (3) There is a notion of Zariski-Riemann spaces for monoids. This is used to prove the existence of equivariant compactification of toric varieties in [EI].

5 The property of the topology generated by ideals

5.1 Preliminary observations

Let V be a self enhancing algebraic type, W the type of commutative monoid objects in V -algebras, and R a W -algebra. So far, we defined the Grothendieck topology \mathcal{O} by the ideals of R , without the reason why. In fact, there is no explicit reason for this choice, if we only aim to construct a natural $(W\text{-}\mathbf{alg})$ -valued space. In addition, we have another canonical way of defining \mathcal{O} : for any W -algebra R , let \mathcal{O}_R be the set of *all* finite sets $\{R_{f_i} \rightarrow R\}_i$ of localizations of finite type, which satisfies the descent datum. Then, we have another $(W\text{-}\mathbf{alg})$ -valued space, which we will denote by $\mathrm{Spec}' R$. We have a immersion $\mathrm{Spec}' R \rightarrow \mathrm{Spec} R$ of $(W\text{-}\mathbf{alg})$ -valued spaces.

When R is a noetherian commutative ring, then the points of $\mathrm{Spec}' R$ correspond to the points of $\mathrm{Spec} R$ of height less than 2 via Krull's Hauptidealsatz ([Mh], Theorem 13.5).

Why don't we consider this, instead of usual Zariski spectrums? To answer this, we must also see R -modules.

Definition 5.1. We endow a fixed coherent topology on $\mathcal{C} = (W\text{-}\mathbf{alg})$, and this induces the spectrum functor $\mathrm{Spec} : \mathcal{C}^{\mathrm{op}} \rightarrow (\mathcal{C}\text{-}\mathbf{Sch})$. Let R be a W -algebra, $X = \mathrm{Spec} R$, and M an R -module.

- (1) The \mathcal{O}_X -module \tilde{M} is defined by the sheafification of $R_f \mapsto M_f$.
- (2) An \mathcal{O}_X -module \mathcal{F} is *quasi-coherent*, if it is isomorphic to \tilde{M} on each affine open subset $\mathrm{Spec} R$ of X , where M is an R -module.

Example 5.2. Let $R = k[x, y]$ be the polynomial ring of two variables over a field k , $X = \mathrm{Spec}' R$, and $M = R/(x, y)$ the residue field at the origin, regarded as an R -module. Then, M induces an \mathcal{O}_X -module \tilde{M} . However, \tilde{M} becomes 0: let U_x, U_y be open subschemes of X corresponding to the localizations R_x, R_y , respectively. Then U_x, U_y covers X , but $\tilde{M} = 0$ on both open subsets.

This phenomenon occurs since the unique support point of M in the usual spectrum $\mathrm{Spec} R$ is of height 2, and it does not exist in X .

We may also recognize that the Grothendieck topology \mathcal{O} defined by the ideals is the coarsest topology such that $M = 0 \Leftrightarrow \tilde{M} = 0$ for any R -module M .

The importance of looking at R -modules instead of R itself is already well known, and typically it appears in the Morita theory:

Theorem 5.3 (Morita). Let V be a self enhancing algebraic type, and R be a monoid object in $(V\text{-}\mathbf{alg})$. Then, we have a natural isomorphism

$$Z(R) \simeq \text{End}(\text{Id}_{(\mathbf{mod}\text{-}R)})$$

of monoid objects in $(V\text{-}\mathbf{alg})$, where $Z(R)$ is the center of R , and $(\mathbf{mod}\text{-}R)$ is the category of right R -modules. In particular, R can be recovered from the category of R -modules when R is commutative.

Proof. In the sequel, any R -module is a right R -module.

Let $F : \text{Id} \rightarrow \text{Id}$ be a natural endomorphism of $\text{Id}_{(\mathbf{mod}\text{-}R)}$. Then $F_R : R \rightarrow R$ is given by a left multiplication by an element $z \in R$, since R is a free R -module generated by one element. Since F is natural, F_R must commute with any left multiplication $a \times (-) : R \rightarrow R$ by an element $a \in R$. This shows that z is in the center of R . The universal property of coproducts tells that $F_P : P \rightarrow P$ is also a left multiplication by z , for any free R -module P .

Let M be an arbitrary R -module. Then we have a surjective homomorphism $P \rightarrow M$ from a free P -module. The commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & M \\ z \times (-) \downarrow & & \downarrow F_M \\ P & \longrightarrow & M \end{array}$$

tells that F_M is also a left multiplication by z . □

Therefore, our next aim is to seek for the coarsest topology \mathcal{O} such that $\tilde{M} = 0$ implies $M = 0$ for arbitrary V and arbitrary commutative monoid object R , and R -module M . Or, in other words, we must verify that the topology induced by ideals is the appropriate one.

When V is not the algebraic type of abelian groups, congruence relations need not be represented by ideals. Therefore, we must replace ideals by congruences. However, we run into trouble when we do this for general cases.

Example 5.4. The typical example is when V is the null algebraic type, namely when an R -module is a set with an action of a monoid R . Then, the set of congruences of R may not have a maximal element, since the set of all congruences is not finitely generated.

This means that, if we try to construct a complete distributive lattice from the congruence directly, then the space may not have enough points.

One possible solution is to pass through other algebraic objects, such as rings, so that we can obtain a distributive lattice from congruences. This idea is realized in [D2].

However, it happens that, looking at ideals is sufficient to detect modules in important cases. In the sequel, the topology \mathcal{O} constructed from ideals as in §4 will simply be called the “ideal topology”.

5.2 The universality of the ideal topology

Let V be a self enhancing algebraic type, with a constant (in other words, 0-ary) operator 0 , and W the type of commutative monoid object of V -algebras. We endow the ideal topology on $\mathcal{C} = (W\text{-}\mathbf{alg})^{\text{op}}$ so that we have the spectrum functor $\text{Spec} : \mathcal{C} \rightarrow (\mathcal{C}\text{-}\mathbf{sch})$.

In this case, we see that the ideal topology is in fact, the coarsest topology distinguishing R -modules.

Proposition 5.5. Let R be any non-zero W -algebra, and \mathfrak{a} be a congruence of R as an R -module.

- (1) $\mathfrak{a} = R \times R$ if and only if $(1, 0) \in \mathfrak{a}$.
- (2) Any non-unital congruence \mathfrak{a} is a subcongruence of a maximal congruence.

Proof. (1) Let $M = R/\mathfrak{a}$ be the quotient R -module. Then for any $x \in M$,

$$x = 1 \cdot x = 0 \cdot x = 0.$$

- (2) This follows from the fact that the set \mathcal{S} of non-unital congruences of R including \mathfrak{a} is an inductively ordered set, since the unit congruence is principally generated by $(1, 0)$.

□

Theorem 5.6. For any W -algebra R and any R -module M , $\tilde{M} = 0$ if and only if $M = 0$.

Proof. Let $x \in M$ be a non-zero element. We have an isomorphism $R/\mathfrak{a} \rightarrow Rx$ of R -modules, where \mathfrak{a} is a congruence on R as an R -module. Since $(1, 0) \notin \mathfrak{a}$, there is a maximal congruence \mathfrak{m} containing \mathfrak{a} . Let \mathfrak{n} be the subset of R consisting of elements a such that $(a, 0) \in \mathfrak{m}$. We see that \mathfrak{n} is a prime ideal, and the stalk $\tilde{M}_{\mathfrak{n}}$ is non-zero. \square

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